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Covariant, Algebraic, and Operator Spinors

V. L. Figueiredo,[1] E. Capelas de Oliveira,[1] and W. A. Rodrigues, Jr.[1]

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We deal with three different definitions for spinors: (I) the *covariant definition*, where a particular kind of *covariant spinor* (c-spinor) is a set of complex variables defined by its transformations under a particular spin group; (II) the *ideal* definition, where a particular kind of algebraic spinor (e-spinor) is defined as an element of a lateral ideal defined by the idempotent e in an appropriated real Clifford algebra $\mathbb{R}_{p,q}$ (when e is primitive we write a-spinor instead of e-spinor); (III) the operator definition where a particular kind of operator spinor (o-spinor) is a Clifford number in an appropriate Clifford algebra $\mathbb{R}_{p,q}$ determining a set of tensors by bilinear mappings. By introducing the concept of "spinorial metric" in the space of minimal ideals of a-spinors, we prove that for $p+q\leq 5$ there exists an equivalence from the group-theoretic point of view among covariant and algebraic spinors. We also study in which sense o-spinors are equivalent to c-spinors. Our approach contain the following important physical cases: Pauli, Dirac, Majorana, dotted, and undotted two-component spinors (Weyl spinors). Moreover, the explicit representation of these c-spinors as a-spinors permits us to obtain a new approach for the spinor structure of space-time and to represent Dirac and Maxwell equations in the Clifford and spin-Clifford bundles over space-time.

1. INTRODUCTION

Three essentially different definitions of spinors appear in the literature.

(I) The covariant definition (Cartan, 1966; Brauer and Weyl, 1935), where a particular kind of covariant spinor (c-spinor) is a set of complex variables defined by its transformations under a particular spin group.

(II) The ideal definition (Chevalley, 1954; Riez, 1958; Graf, 1978), where a particular kind of an algebraic spinor (e-spinor) is an element of a lateral ideal (defined by the idempotent e) in an appropriate Clifford algebra. (When e is primitive idempotent we write a-spinor instead of e-spinor.)

[1]Instituto de Matemática, Estatística e Ciência da Computação, IMECC-UNICAMP, 13081, Campinas, S. P., Brazil.

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(III) The operator definition (Hestenes and Sobezyk, 1984), where a particular kind of operator spinor (*o*-spinor) is a Clifford number in an appropriate Clifford algebra $\mathbb{R}_{p,q}$ determining a set of tensors by bilinear mappings (see Section 2 for notation).

The so-called pure spinors recently used by Caianiello (1988) and Budinich and Trautman (1986) are special cases of *c*-spinors (or *e*-spinors) and will not be analyzed in this paper. From the point of view of this paper they are not as fundamental as is usually thought.

The usual presentation of *e*-spinors as elements of lateral ideals in Clifford algebras (Chevalley, 1954; Hestenes and Sobezyk, 1984; Graf, 1978) as well as the introduction in this context of the groups $Spin_+(p, q)$ do not leave clear the relation among these objects and the *c*-spinors and the universal covering groups of some groups $SO_+(p, q)$ used in theoretical physics. The same is true in relation to *o*-spinors.

The main purpose of the present paper is to clear up the situation, and in the process we obtain very interesting results. In particular, we prove that all the *c*-spinors used by physicists can be represented by appropriate *e*-spinors. From the explicit construction of the *e*-spinors (representing *c*-spinors) by the "idempotent method" (Section 2.3), we see that these objects are nothing more than the sum of multivectors (or multiforms).

This result reveals the hidden geometrical nature of spinors. It shows that, contrary to usual claims (Budinich and Trautman, 1986; Crumeyrolle, 1969, 1971; Bugajska, 1979; Penrose and Rindler, 1984), spinors are not more fundamental than tensors. Also, it shows the limited validity of the geometrical interpretations already proposed in the literature (Bugajska, 1979; Penrose and Rindler, 1984; Hestenes, 1967, 1971*a*,*b*, 1975; Lounesto, 1986; Santaló, 1976) for Pauli *c*-spinors, Weyl *c*-spinors (i.e., two-component dotted and undotted *c*-spinors), Dirac *c*-spinors, and Majorana *c*-spinors.

To formulate our problem, we start by recalling the kinds of *c*-spinors used by physicists.

1.1. Pauli *c*-Spinors

These are the vectors of a complex 2-dimensional space $\mathbb{C}^2$ equipped with the spinorial metric

$$\beta_p: \quad \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}, \qquad \beta_p(\psi, \varphi) = \psi^* \varphi \tag{1}$$

$$\psi = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \qquad \varphi = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \qquad z_i, y_i \in \mathbb{C}, \quad i = 1, 2, \qquad \varphi^* = (\bar{z}_1, \bar{z}_2)$$

where in this text $\bar{z}$ always means the complex conjugate of $z \in \mathbb{C}$.

The spinorial metric is invariant under the action of the group $SU(2)$, i.e., if $u \in SU(2)$, then $\beta_p(u\psi, u\varphi) = \beta_p(\psi, \varphi)$. As it is well known, Pauli *c*-spinors carry the fundamental (irreducible) representation $D^{1/2}$ of $SU(2)$ (Miller, 1972; Landau and Lifschitz, 1971).

1.2. Weyl *c*-Spinors

These objects were introduced by Weyl (1929) and called by van der Waerden (1932) undotted and dotted two-component spinors. We have the following definitions.

1.2.1. Contravariant Undotted Spinors

These are the elements of a complex 2-dimensional space $\mathbb{C}^2$ equipped with the spinorial metric

$$\beta: \quad \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}; \qquad \beta(\eta, \xi) = \eta^t \mathbf{C} \xi \tag{2}$$

$$\eta = \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The spinorial metric β is invariant under the action of the group $SL(2, \mathbb{C})$, i.e., if $\eta \mapsto u\eta$, $\xi \mapsto u\xi$, then

$$\beta(\eta, \xi) = \beta(u\eta, u\xi) \leftrightarrow u^t \mathbf{C} u = \mathbf{C} \leftrightarrow u \in SL(2, \mathbb{C}) \tag{3}$$

1.2.2. Covariant Undotted Spinors

These are the elements of the dual space $\overset{\Delta}{\mathbb{C}}{}^2$ defined by

$$\overset{\Delta}{\mathbb{C}}{}^2 \ni \overset{\Delta}{\eta}: \quad \mathbb{C}^2 \to \mathbb{C}; \qquad \overset{\Delta}{\eta}(\xi) = \overset{\Delta}{\eta}\xi = \beta(\eta, \xi) \tag{4}$$

It follows that

$$\overset{\Delta}{\eta} = \eta^t \mathbf{C} = (\eta_1, \eta_2) = (\eta^2, -\eta^1) \tag{5}$$

The transformation law of the covariant undotted spinors that leaves the spinorial metric invariant under $SL(2, \mathbb{C})$ is then

$$\overset{\Delta}{\eta} \mapsto \overset{\Delta}{\eta} u^{-1}, \qquad u \in SL(2, \mathbb{C}) \tag{6}$$

1.2.3. Contravariant Dotted Spinors

These are the elements of the space $\dot{\mathbb{C}} \equiv (\mathbb{C}^2)^*$, i.e., $\dot{\mathbb{C}}^2 \ni \dot{\eta} = (\eta^{\dot{1}}, \eta^{\dot{2}}) \equiv (\bar{\eta}^1, \bar{\eta}^2) = \eta^*$, $\eta \in \mathbb{C}^2$ equipped with the spinorial metric $\dot{\beta}$,

$$\dot{\beta}: \quad \dot{\mathbb{C}}^2 \times \dot{\mathbb{C}}^2 \to \mathbb{C}, \qquad \dot{\beta}(\dot{\eta}, \dot{\xi}) = \dot{\eta} \mathbf{C} \dot{\xi}^t \tag{7}$$

and we have that

$$\dot{\beta}(\dot{\eta}, \dot{\xi}) = \dot{\beta}(\dot{\eta} u^*, \dot{\xi} u^*) \leftrightarrow u^* \mathbf{C} u^{*t} = \mathbf{C} \tag{8}$$

1.2.4. Covariant Dotted Spinors

These are the element of the dual space $\overset{\Delta}{\dot{\mathbb{C}}}{}^2$, defined by

$$\overset{\Delta}{\dot{\mathbb{C}}}{}^2 \ni \overset{\Delta}{\dot{\xi}} = \beta(\cdot, \dot{\xi}); \qquad \dot{\eta}(\overset{\Delta}{\dot{\xi}}) \equiv \dot{\eta}\overset{\Delta}{\dot{\xi}} = \dot{\eta}\mathbf{C}\dot{\xi}^t \tag{9}$$

It follows that

$$\overset{\Delta}{\dot{\xi}} = \mathbf{C}\dot{\xi}^t = \begin{pmatrix} \xi_{\dot{1}} \\ \xi_{\dot{2}} \end{pmatrix} \equiv \begin{pmatrix} \xi^{\dot{2}} \\ -\xi^{\dot{1}} \end{pmatrix} \equiv \begin{pmatrix} \bar{\xi}^2 \\ -\bar{\xi}^1 \end{pmatrix} \tag{10}$$

It is clear that the laws of transformations of the dotted spinors under the action of $SL(2, \mathbb{C})$ are

$$\dot{\eta} \mapsto \dot{\eta}u^*; \qquad \overset{\Delta}{\dot{\eta}} \to (u^*)^{-1}\overset{\Delta}{\dot{\eta}} \tag{11}$$

The matrices u and $(u^*)^{-1}$ are the (nonequivalent) representations $D^{(1/2,0)}$ and $D^{(0,1/2)}$ of $SL(2, \mathbb{C})$.

1.3. Dirac *c*-Spinors

These are the vectors of a complex 4-dimensional space $\mathbb{C}^4$ equipped with the spinorial metric (Landau and Lifschitz, 1971; Srivastrava, 1974)

$$\beta_d: \quad \mathbb{C}^4 \times \mathbb{C}^4 \to \mathbb{C}, \qquad \beta_d = (\psi_d, \phi_d) = \psi_d^t B \phi_d$$

where a Dirac *c*-spinor $\psi_d(\phi_d)$ is defined as

$$\mathbb{C}^2 \oplus \overset{\Delta}{\dot{\mathbb{C}}}{}^2 = \mathbb{C}^4 \ni \psi_d = \eta + \overset{\Delta}{\dot{\xi}} = \begin{pmatrix} \eta^1 \\ \eta^2 \\ \xi_{\dot{1}} \\ \xi_{\dot{2}} \end{pmatrix} \tag{12}$$

In the canonical basis of $\mathbb{C}^4$ the matrix B is the representation of β_d and we have

$$B = \begin{pmatrix} \mathbf{C} & 0 \\ 0 & \mathbf{C} \end{pmatrix} \tag{13}$$

$$\beta_d(\psi_d, \phi_d) = \beta_d(\rho(u)\psi_d, \rho(u)\phi_d)$$

$$\rho(u) = \begin{pmatrix} u & 0 \\ 0 & (u^*)^{-1} \end{pmatrix}, \qquad u \in SL(2, \mathbb{C}) \tag{14}$$

The transformation law of the Dirac *c*-spinors is then

$$\psi_d \mapsto \begin{bmatrix} u & 0 \\ 0 & (u^*)^{-1} \end{bmatrix} \psi_d \tag{15}$$

which means that Dirac *c*-spinors, as is well known, carry the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2, \mathbb{C})$.

1.4. Standard Dirac c-Spinors

If $\rho(u)$ is a representation of $SL(2, \mathbb{C})$, then $S\rho(u)S^{-1}$, with $SS^{-1} = S^{-1}S = 1$ is also a representation. Under a similarity transformation the spinor $\psi_d \mapsto S\psi_d$, which in general mixes the components of $\mathbb{C}^2$ with those of $\overset{\Delta}{\dot{\mathbb{C}}}{}^2$. A particular mixing is convenient in writing Dirac's equation. We define standard Dirac spinors as the objects ψ_s such that

$$\mathbb{C}^4 \ni \psi_s = \begin{pmatrix} \phi \\ \lambda \end{pmatrix} \tag{16}$$

where

$$\phi = \frac{1}{\sqrt{2}}(\xi + \overset{\Delta}{\dot\eta}); \qquad \lambda = \frac{1}{\sqrt{2}}(\xi - \overset{\Delta}{\dot\eta})$$

where $\xi \in \mathbb{C}^2$ and $\overset{\Delta}{\dot\eta} \in \overset{\Delta}{\dot{\mathbb{C}}}{}^2$ and the sums in ϕ and λ are in the sense of sums of complex numbers for each component. It is well known that ψ_d and ψ_s are related by a unitary transformation ($S^{-1} = S^*$) which leave unchanged the bilinear covariant constructed from ψ_d and ψ_d^* (Landau and Lifschitz, 1971; Srivastrava, 1974).

1.5. Outline of This Work

We now ask the main question to which this paper is addressed: to which Clifford algebras are the c-spinors described in Sections 1.1–1.4 to be associated?

We are going to give an original answer to the above question by introducing a natural scalar product (see Section 3) in certain lateral ideals of certain *real* Clifford algebras that "mimic" what has been described in Sections 1.1–1.4. To this end, in Section 2 we give the main properties of Clifford algebras over the reals (Chevalley, 1954; Riesz, 1958; Porteous, 1981; Atiyah *et al.*, 1964; Blaine Lawson and Michelsohn, 1983; Felzenswalb, 1979; Coquereaux, 1982; Salingaros and Wene, 1985; Micali, 1986). The material presented fixes our notation and is the minimum necessary to permit the formulation of our ideas in a rigorous way. We follow with minor modifications the notations of Porteous (1981).

In Section 3 we define the a-spinors as elements of minimal lateral ideals and the e-spinors are the elements of lateral ideals (*not necessarily minimal*) in real Clifford algebras. The a-spinors or e-spinors of each one of the Clifford algebras *studied* in this paper has a natural right F-linear space structure over one of the fields $F = \mathbb{R}$ or $\mathbb{C}$, or $\mathbb{H}$, respectively the real, complex, and quaterniom fields (Section 2).

We introduce for each a-spinor space $I = \mathbb{R}_{p,q}e$ a *natural scalar product* (spinorial metric), i.e., a nondegenerate bilinear mapping Γ: $I \times I \to F$, where F is the natural scalar field associated with the vector structure of $I \subset \mathbb{R}_{p,q}$.

Our approach to the natural scalar product shows for $p+q \leq 5$ the groups $Spin_+(p, q)$ are the groups that leave the spinorial metric invariant. Thus, our approach to the scalar product is different from the one discussed by Lounesto (1981) and, as we shall see, offers a solution for the main question formulated above.

In Section 4 we analyze in detail the special cases $SU(2) \simeq Spin(3,0)$ and $SL(2, \mathbb{C}) \simeq Spin_+(1,3)$ and identify respectively the ideals that contain the objects corresponding to Pauli c-spinors in $\mathbb{R}_{3,0}$ and the Weyl c-spinors and Dirac c-spinors in $\mathbb{R}_{1,3}$ (the space-time algebra) and $\mathbb{R}_{3,1}$ (the Majorana algebra). Our identifications are all based on explicit proofs that the representative space of a-spinors (or e-spinors) of each one of the c-spinors mentioned above carries the correct representation of the corresponding *spin* group (according to the theory of group representation). We show also that the original Dirac algebra $\mathbb{C}(4)$ must be identified for physical reasons with the real Clifford algebra $\mathbb{R}_{4,1}$.

Now, it is well known that physical theories use spinor fields. Indeed, there are theories which use c-spinor fields (Bleecker, 1981; Lichnerowicz, 1964) and theories that use e-spinor fields (Graf, 1978; Benn and Tucker, 1983*a*,*b*, 1985*a*,*b*; Blau, 1987). The c-spinor fields are sections of the so-called covariant spinor bundle and the e-spinor fields are elements of the Clifford or the spin-Clifford bundles. These two bundles are of very different nature. Indeed, the Clifford bundle always exists (Graf, 1978), whereas the existence of the covariant spinor bundle imposes several constraints on the base manifold of the bundle [which is taken as a Lorentzian manifold modeling space-time (Crumeyrolle, 1969, 1971; Milnor, 1963; Bichteler, 1963; Geroch, 1968; Rodrigues and Figueiredo, 1989, 1990*a*)].

The use of algebraic spinors fields (representing the covariant Dirac spinor fields) taken as ideal sections of the Clifford bundle in Graf (1978), Benn and Tucker (1983*a*,*b*, 1985*a*,*b*), and Blau (1987) shows that the local representation of these a-spinor fields does not transform under local Lorentz transformations in the same way as the c-spinor fields.

In Rodrigues and Figueiredo (1990*a*) (henceforth called II), we study the structure of the covariant spinor bundle, the Clifford bundle, and a new bundle called the spin-Clifford bundle. We show that to obtain an a-spinor field with the same transformation law as a c-spinor field under a local Lorentz transformation, it is necessary to take the a-spinor field as a section of the spin-Clifford bundle. In II we also use these bundles together with the explicit construction of the Weyl a-spinor obtained in this paper (Section 4.2) to study the spinor structure of space-time and to obtain a new proof of Geroch's (1968) theorem that requires only elementary knowledge of the bundle reduction process.

The definition of o-spinors given by Hestenes for $\mathbb{R}_{1,3}$ and generalized by Dimakis (1986) for real Clifford algebras $\mathbb{R}_{p,q}$ is presented in Section 4.6 together with their relations with c-spinors and e-spinors.

In Rodrigues and de Oliveira (1990*b*) (henceforth called III) we show how to write Dirac and Maxwell equations in the Clifford and spin-Clifford bundles over space-time and study in detail the transformation laws of these fields viewed as sections of the Clifford or the spin-Clifford bundle. Finally, in Section 5 we present our conclusions.

2. SOME GENERAL FEATURES OF CLIFFORD ALGEBRAS

2.1. Introduction

Let V be a vector space of finite dimension n over the field F and let Q be a nondegenerate quadratic form on V. The Clifford algebra $C(V, Q) = T(V)/I_Q$, where $T(V)$ is the tensor algebra of $V(T(V) = \sum_{i=1}^{\infty} T^i(V)$; $T^{(0)}(V) = F$; $T^1(V) = V$; $T^r(V) = \otimes^r V)$ and I_Q is the bilateral ideal generated by the elements of the form $x \otimes x - Q(x)1$, $x \in V$. The signature of Q is arbitrary. The Clifford algebra so constructed is an associative algebra with unit. The space V is naturally imbedded in $C(V, Q)$. We have

$$V \overset{i}{\hookrightarrow} T(V) \overset{j}{\rightarrow} T(V)/I_Q = C(V, Q), \quad i_Q = j \circ i; \qquad V \equiv I_Q(V) \subset C(V, Q)$$

Let $C^+(V, Q)$ [respectively $C^-(V, Q)$] be the j-image of $\sum_{i=0}^{\infty} T^{2i}(V)$ [respectively $\sum_{i=0}^{\infty} T^{2i+1}(V)$] in $C(V, Q)$. The elements of $C^+(V, Q)$ form a subalgebra of $C(V, Q)$ called the even subalgebra of $C(V, Q)$.

$C(V, Q)$ has the following universal property: If A is an associative F-algebra with unit, then all linear mappings ϕ: $V \rightarrow A$ such that $(\phi(x))^2 = Q(x)1$, $\forall x \in V$, can be extended in a unique way to a homomorphism ϕ: $C(V, Q) \rightarrow A$.

In $C(V, Q)$ there exist three linear mappings which are quite natural. They are extensions of the mappings:

(a) *Main Involution.* An automorphism $^{\square}$: $C(V, Q) \rightarrow C(V, Q)$ extension of α: $V \rightarrow T(V)/I_Q$, $\alpha(x) = -i_Q(x) = -x$, $\forall x \in V$.

(b) *Reversion.* An antiautomorphism $*$: $C(V, Q) \rightarrow C(V, Q)$ extension of $'$: $T^r(V) \rightarrow T^r(V)$, $T^r(V) \ni x = x_{i_1} \otimes \cdots \otimes x_{i_r} \rightarrow x' = x_{i_r} \otimes \cdots \otimes x_{i_1}$.

(c) *Conjugation.* $\tilde{}$: $C(V, Q) \rightarrow C(V, Q)$, defined by the composition of the automorphism $^{\square}$ with the antiautomorphism $*$, i.e., if $x \in C(V, Q)$, then $\tilde{x} = (x^*)^{\square}$.

$C(V, Q)$ can be described through its generators, i.e., if $\{e_i\}$, $i = 1, 2, \ldots, n$, is a Q-orthonormal basis of V, then $C(V, Q)$ is generated by 1 and the e_i' subject to the conditions $e_ie_i = Q(e_i)1$ and $e_ie_j + e_je_i = 0$, $i \neq j$, $i, j = 1, 2, \ldots, n$. If V is an n-dimensional real vector space, then we can choose a basis $\{e_i\}$ for V such that $Q(e_i) = \pm 1$.

2.2. The Real Clifford Algebras $\mathbb{R}_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $p + q = n$ equipped with a metric $g: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to R$. Let $\{e_i\}$ be the canonical basis of $\mathbb{R}^{p,q}$ such that

$$g(e_i, e_j) = g_{ij} = g(e_j, e_i) = g_{ji} = \begin{cases} +1, & i = j = 1, 2, \ldots, p \\ -1, & i = j = p+1, \ldots, p+q = n \\ 0, & i \neq j \end{cases}$$

The Clifford algebra $\mathbb{R}_{p,q} = C(\mathbb{R}^{p,q}, Q)$; $p + q = n$, in the Clifford algebra over the real field $\mathbb{R}$, generated by 1 and the $\{e_i\}$, $i = 1, \ldots, n$, such that $Q(e_i) = g(e_i, e_i)$. The $\mathbb{R}_{p,q}$ is obviously of dimension 2^n and it is the direct sum of the vector spaces $\mathbb{R}^k_{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical base for $\mathbb{R}^k_{p,q}$ are the elements $e_A = e_{\alpha_1} \cdots e_{\alpha_k}$, $1 \leq \alpha_1 \leq \cdots \leq \alpha_k \leq n$. The element $e_J = e_1 \cdots e_n \in \mathbb{R}^n_{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $e_1, \ldots, e_n$ in $\mathbb{R}^1_{p,q} = \mathbb{R}^{p,q}$. The center of $\mathbb{R}_{p,q}$ is $\mathbb{R}^0_{p,q} = \mathbb{R}$ if n is even and it is the direct sum $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$ if n is odd (Blaine Lawson and Michelsohn, 1983; Felzenswalb, 1979). All Clifford algebras are semisimple. If $p + q = n$ is even, $\mathbb{R}_{p,q}$ is a simple algebra and if $p + q = n$ is odd, we have the following possibilities:

(a) $\mathbb{R}_{p,q}$ is simple $\leftrightarrow e_J^2 = -1 \leftrightarrow p - q \neq 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to C.
(b) $\mathbb{R}_{p,q}$ is not simple $\leftrightarrow e_J^2 = +1 \leftrightarrow p - q = 1 \pmod 4 \leftrightarrow$ center $\mathbb{R}_{p,q}$ is isomorphic to $\mathbb{R}^0_{p,q} \oplus \mathbb{R}^n_{p,q}$.

From the fact that all semisimple algebras are the direct sum of two simple algebras (Blaine Lawson and Michelson, 1983) and from *Weddenburn's Theorem* [if A is a simple algebra, then A is equivalent to $F(m)$, where F is a division algebra and m and F are unique (modulo isomorphisms)], we obtain from the point of view of representation theory $\mathbb{R}_{p,q} \simeq F(m)$ or $\mathbb{R}_{p,q} \simeq F(m) \oplus F(m)$, where $F(m)$ is the matrix algebra of dimension $m \times m$ (for some m) with coefficients in $F = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Table I (where $[n/2]$ means the integral part of $n/2$) presents the representation of $\mathbb{R}_{p,q}$ as a matrix algebra (Blaine Lawson and Michelsohn, 1983; Coquereaux, 1982).

Table I. Representation of the Real Clifford Algebra $\mathbb{R}_{p,q}$ As a Matrix Algebra

$p-q \pmod 8$	0	1	2	3	4	5	6	7
$\mathbb{R}_{p,q}$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]})$	$\mathbb{R}(2^{[n/2]})$	$\mathbb{C}(2^{[n/2]})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{R}(2^{[n/2]-1}) \oplus \mathbb{H}(2^{[n/2]-1})$	$\mathbb{H}(2^{[n/2-1]})$	$\mathbb{C}(2^{[n/2]})$

2.3. Minimal Lateral Ideals of $\mathbb{R}_{p,q}$

The minimal left ideals of a semisimple algebra A are of the type Ae, where e ($e^2 = e$) is a primitive idempotent of A. An idempotent is primitive if it cannot be written as a sum of two nonzero orthogonal idempotents, i.e., $e \neq \hat{e} + \check{e}$, where $\hat{e}^2 = \hat{e}$, $\check{e}^2 = \check{e}$, and $\hat{e}\check{e} = \check{e}\hat{e} = 0$ (Blaine Lawson and Michelsohn, 1983). Recall that when $p + q = n$ is even, $\mathbb{R}_{p,q} \simeq F(m)$ (Table I). We also have the following result.

Theorem. The maximum number of pairwise orthogonal idempotents in $F(m)$ is m (Felzenswalb, 1979).

The decomposition of $\mathbb{R}_{p,q}$ into minimal ideals is then characterized by a *spectral set* $\{e_{pq,i}\}$ of idempotent elements of $\mathbb{R}_{p,q}$ such that:

(a) $\sum e_{pq,i} = 1$.
(b) $e_{pq,i}e_{pq,j} = \delta_{ij}e_{pq,i}$.
(c) The rank of $e_{pq,i}$ is minimal and nonzero; i.e., $e_{pq,i}$ is primitive.

Here the rank of $e_{pq,i}$ is defined as the rank of the $\oplus \Lambda^d(R^{p,q})$-morphism $e_{pq,i}: \psi \to \psi e_{pq,i}$, where $\oplus\ \Lambda^d(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $\mathbb{R}_{p,q} = \sum I^i_{p,q}$, $I^i_{p,q} = \mathbb{R}_{p,q}e_{pq,i}$, and $\psi \in I^i_{p,q} \subset \mathbb{R}_{p,q}$ is such that $\psi e_{pq,i} = \psi$. Conversely, any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi e_{pq,i} = \psi$.

We have the following theorem.

Theorem (Lounesto, 1981). A minimal left ideal of $\mathbb{R}_{p,q}$ is of the type $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$, where $e_{pq} = 1/2(1 + e_{\alpha_1}) \cdots 1/2(1 + e_{\alpha_k})$ is a primitive idempotent of $\mathbb{R}_{p,q}$ and where $e_{\alpha_1}, \ldots, e_{\alpha_k}$ is a set of commuting elements of the canonical basis of $\mathbb{R}_{p,q}$ such that $(e_{\alpha_1})^2 = 1$, $i = 1, \ldots, k$, that generates a group of order $k = q - r_{q-p}$ and r_i are the Radon–Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and Table II.

Table II. Radon–Hurwitz Number

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

If we have a linear mapping $L_a : \mathbb{R}_{p,q} \to \mathbb{R}_{p,q} \ni L_a(x), \forall x \in \mathbb{R}_{p,q}$, and where $a \in \mathbb{R}_{p,q}$, then since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $\mathbb{R}_{p,q}$, we can consider $L_a|_{I_{p,q}} : I_{p,q} \to I_{p,q}$. We have the following result.

Theorem. If $p+q=n$ is even or odd with $p-q \neq 1 \pmod 4$, then

$$\mathbb{R}_{p,q} \simeq \mathscr{L}_F(I_{p,q}) \simeq F(m) \tag{17}$$

where $F \simeq \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, $\mathscr{L}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field F, $m = \dim_F(I_{p,q})$, and $F \simeq eF(m)e$, e being the representation of e_{pq} in $F(m)$. If $p+q=n$ is odd, with $p-q=1 \pmod 4$, then $\mathbb{R}_{p,q} \simeq \mathscr{L}_F(I_{p,q}) \simeq F(m) \oplus F(m)$, $m = \dim_F(I_{p,q})$, and $e_{pq}\mathbb{R}_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

With the above isomorphisms we can identify the minimal left ideals $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$.

Now, with the ideas introduced above it is a simple exercise to find a primitive idempotent of $\mathbb{R}_{p,q}$. We have the following algorithm. We first look at Table I and find the matrix algebra to which our particular $\mathbb{R}_{p,q}$ is isomorphic. Let $\mathbb{R}_{p,q} \simeq F(m)$ for a particular F and m.[2] Next we take from the canonical basis $\{e_A\}$ of $\mathbb{R}_{p,q}$

$$e_A = e_{\beta_1} \cdots e_{\beta_k}, \qquad 1 \leq \beta_1 \leq \cdots \leq \beta_k \leq n, \qquad p+q=n$$

an element $e_{\alpha_1} \in \{e_A\}$ such that $e_{\alpha_1}^2 = 1$. We then construct the idempotent $e_{pq} = 1/2(1+e_{\alpha_1})$ and calculate $\dim_F(I_{p,q})$. If $\dim_F(I_{p,q}) = m$, then e_{pq} is primitive. If $\dim_F(I_{p,q}) \neq m$, then choose[3] $\{e_A\} \ni e_{\alpha_2} | e_{\alpha_2}^2 = 1$ and construct the idempotent $e'_{pq} = 1/2(1+e_{\alpha_1})1/2(1+e_{\alpha_2})$ and calculate $\dim_F(I'_{p,q})$, where $I'_{p,q} = \mathbb{R}_{p,q}e'_{pq}$. If $\dim_F(I'_{p,q}) = m$, then e_{pq} is primitive. Otherwise, repeat the procedure. According to the theorem above, the process is finite.

We will discuss the problem of the *equivalence* of representations of $\mathbb{R}_{p,q}$ when we take the minimal left ideals (instead of some vector space isomorphic to them) as representation modules of $\mathbb{R}_{p,q}$, after the introduction of the concept of the spin groups (Section 3).

3. ALGEBRAIC SPINORS, SPIN GROUP, SPINORIAL REPRESENTATION, AND SPINORIAL METRIC

3.1. The Spin Group *Spin* (p, q)

The invertible elements $u \in \mathbb{R}_{p,q}$ such that $\forall x \in \mathbb{R}^1_{p,q} \equiv \mathbb{R}^{p,q}$ we have $uxu^{-1} \in \mathbb{R}^1_{p,q}$ form a multiplicative group of $\mathbb{R}_{p,q}$ called $\Gamma(p, q)$. Consider

[2] We are supposing $\mathbb{R}_{p,q}$ is simple. The procedure is also straightforward when $\mathbb{R}_{p,q}$ is semisimple.
[3] All elements e_{α_1} are actual commuting elements as stated in the last theorem.

now the norm mapping $N: \mathbb{R}_{p,q} \to \mathbb{R}_{p,q}$, $N(u) = \tilde{u}u$. If $u \in \Gamma(p, q)$, then N is a homomorphism of the group $\Gamma(p, q)$ into the multiplicative group of the nonnull multiples of $\mathbb{R}_{p,q}$.

We define the groups $Pin(p, q) = \{u \in \Gamma(p, q) \mid N(u) = \pm 1\}$, $Spin(p, q) = Pin(p, q) \cap \mathbb{R}^+_{p,q}$, and $Spin_+(p, q) = \{u \in \Gamma(p, q) \mid \tilde{u}u = +1\} \cap \mathbb{R}^+_{p,q}$ as the connected component of $Spin(p,q)$ that contains the identity. We can show that $Spin_+(p, q)/Z_2 = SO_+(p, q)$, where $SO_+(p, q)$ is the special rotation group of $\mathbb{R}^{p,q}$. For more details see Figueiredo *et al.* (1988).

3.2. Algebraic Spinors

Given a real Clifford algebra $\mathbb{R}_{p,q}$, we call elementary algebraic spinors, or *a*-spinors for short, the elements of the minimal left (right) ideal $\mathbb{R}_{p,q}e_{pq}$ ($e_{pq}\mathbb{R}_{p,q}$) or $\mathbb{R}^+_{p,q}e'_{p,q}$ ($e'_{pq}\mathbb{R}^+_{p,q}$), where e_{pq}, $e'_{p,q}$ are primitive idempotents of $\mathbb{R}_{p,q}$. We call algebraic spinors or *e*-spinors the elements of left (right) nonminimal ideals of $\mathbb{R}_{p,q}$.

3.3. Spinorial Representation. Scalar Product of Spinors. The Spinorial Metric

Given the definitions of the group $Spin_+(p, q)$ and of algebraic spinors, we can make the ideals $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ into spinorials of $SO_+(p, q)$ in the sense of group, by introducing a spinorial metric in $I_{p,q}$ that "mimics" the results of Sections 1.1–1.4 and which is invariant under the mappings $\psi \mapsto u\psi$, $u \in Spin_+(p, q)$, $\psi \in I_{p,q}$.

The "transformation law" $\psi \mapsto u\psi$ corresponds to the usual transformation of covariant spinors, but this transformation law is not the one to which an algebraic spinor field is subject when intended as a section of the Clifford bundle. This transformation law is, however, the one to be used when algebraic spinor fields are viewed as sections of the spin-Clifford bundle. All these points that have to do with the equivalence of the minimal ideals of $\mathbb{R}_{p,q}$ as representation modules of $\mathbb{R}_{p,q}$ are discussed in detail in Rodrigues and Figueiredo (1990), Rodrigues and de Oliveira (1990), and Figueiredo *et al.* (1988).

In Section 2.3 we saw that if $\mathbb{R}_{p,q}$ is simple, a minimal left ideal $I_{p,q}$ of $\mathbb{R}_{p,q}$ is of the form $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$, where e_{pq} is a primitive idempotent of $\mathbb{R}_{p,q}$ and $F \simeq e_{pq}\mathbb{R}_{p,q}e_{pq}$ with $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, depending on $p - q = 0, 1, 2 \pmod 8$, $p - q = 3, 7 \pmod 8$, or $p - q = 4, 5, 6 \pmod 8$, respectively (Table I). We can then define an action F in $I_{p,q}$, $F \times I_{p,q} \to I_{p,q}$, by $F \times I_{p,q} \ni (\alpha, \psi) \to \alpha\psi \in I_{p,q}$. In this way $I_{p,q}$ has a natural linear vector space structure over the field F, whose elements are the natural "scalars" of the vector space $I_{p,q}$.

These remarks suggest that we search for a "*natural scalar product*" on $I_{p,q}$, i.e., a nondegenerate bilinear mapping $\Gamma: I_{p,q} \times I_{p,q} \to F$. To this end, we observe that if f and g are F-endomorphisms in $\mathbb{R}_{p,q}$, then we can define a bilinear mapping Γ in $\mathbb{R}_{p,q}$ using f and g. We simply take $\Gamma(\psi, \varphi) = f(\psi)g(\varphi)$, $\psi, \varphi \in \mathbb{R}_{p,q}$. Considering that $I_{p,q} = \mathbb{R}_{p,q}e_{pq}$ has a natural structure of vector space over F, we can take the restriction of Γ to $I_{p,q}$, and ask the following question:

For $\psi, \varphi \in I_{p,q}$, when does $\Gamma(\psi, \varphi) \in F$?

As we saw in Section 2.1, we have three natural isomorphisms defined in $\mathbb{R}_{p,q}$, the main involution, the reversion, and the conjugation, denoted, respectively, by $^{\square}$, *, and $\tilde{}$. Combining these isomorphisms with the identity mapping, we can define the following bilinear mappings:

$$\begin{aligned} &\Gamma_i: \quad I_{p,q} \times I_{p,q} \to \mathbb{R}_{p,q}, \qquad i = 1, 2, 3 \\ &\Gamma_1(\psi, \varphi) = \psi^{\square}\varphi \\ &\Gamma_2(\psi, \varphi) = \psi^*\varphi \\ &\Gamma_3(\psi, \varphi) = \tilde{\psi}\varphi, \qquad \forall \psi, \varphi \in I \end{aligned} \tag{18}$$

As already observed in Section 2.1, the main involution is an automorphism, whereas the reversion and conjugation are antiautomorphisms. An automorphism (antiautomorphism) transforms an element of a minimal left ideal into an element of a minimal left ideal (minimal right ideal).

To see the validity of these statements it is enough to observe that the image of a primitive idempotent under an isomorphism is a primitive idempotent and that if $\psi \in I_{p,q} = \mathbb{R}_{p,q}e_{pq}$, then $\psi = xe_{pq}$ with $x \in \mathbb{R}_{p,q}$ and

$$\begin{aligned} \psi^{\square} &= (xe_{pq})^{\square} = x^{\square}e_{pq}^{\square} \Rightarrow \psi^{\square} \in I'_{p,q} = \mathbb{R}_{p,q}e_{pq}^{\square} \\ \psi^* &= (xe_{pq})^* = e_{pq}^* x^* \Rightarrow \psi^* \in {}^*I_{p,q} = e_{pq}^*\mathbb{R}_{p,q} \\ \tilde{\psi} &= (xe_{pq})^{\sim} = \tilde{e}_{pq}\tilde{x} \Rightarrow \tilde{\psi} \in \tilde{I}_{p,q} = \tilde{e}_{pq}\mathbb{R}_{p,q} \end{aligned} \tag{19}$$

Using the isomorphism $\mathbb{R}_{p,q} \simeq \mathscr{L}_F(I_{p,q}) \simeq F(m)$, $m = \dim_F(I_{p,q})$ (when $\mathbb{R}_{p,q}$ is simple; cf. Section 2.3), we identify the elements of the minimal left ideals of $\mathbb{R}_{p,q}$ with the column matrices of $F(m)$. Then, if $\psi \in I_{p,q}$ has a representation as a column matrix of $F(m)$, then ψ^* and $\tilde{\psi}$ have representations as row matrices of $F(m)$, and we get that $\psi^*\varphi$ and $\tilde{\psi}\varphi$ are elements of F.

We identify the scalars of the vector structure of $I_{p,q}$ with multiples of

$$e_{pq} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{20}$$

i.e., as matrices in $F(m)$ multiples of the matrix in equation (20). Sometimes it may be convenient to choose the 1 in $e_{p,q}$ in another line. Through isomorphisms of $\mathbb{R}_{p,q}$ (multiplication by a convenient invertible element $u \in \mathbb{R}_{p,q}$), we can transport $\psi^* \varphi$ or $\psi\varphi$ to the position (1, 1) in the matrix representation of these operations. We then conclude that the *natural scalar products* in $I_{p,q}$ are

$$\beta_i = I_{p,q} \times I_{p,q} \to F, \qquad i = 1, 2 \tag{21}$$

$\beta_1(\psi, \varphi) = u'\psi^*\varphi$ and $\beta_2(\psi, \varphi) = u\tilde{\psi}\varphi$, $\forall \psi, \varphi \in I_{p,q}$ and $u, u' \in \mathbb{R}_{p,q}$, are convenient invertible elements.

Lounesto (1981) obtains the scalar products in equation (21) using similar arguments and immediately proceeds to the classification of the group of automorphisms of these scalar products, i.e., the homomorphisms of F-modules, $I_{p,q} \to I_{p,q}$, $\psi \to s\psi$, $s \in \mathbb{R}_{p,q}$, which preserve the products in equation (18). Observe that from $\beta_1(s\psi, s\varphi) = \beta_1(\psi, \varphi)$ we get $s^*s = 1$ and from $\beta_2(s\psi, s\varphi) = \beta_2(\psi, \varphi)$ we get $\tilde{s}s = 1$ ($\psi, \varphi \in I_{p,q}$). Lounesto calls $G_1 = \{s \in \mathbb{R}_{p,q};\, s^*s = 1\}$, $G_2 = \{s \in \mathbb{R}_{p,q},\, \tilde{s}s = 1\}$.

So in Lounesto's paper there does not appear clearly any relationship between the groups $Spin_+(p, q)$ and the groups G_1 and G_2, with the consequence that we do not have a clear basis to mimic within the Clifford algebras $\mathbb{R}_{p,q}$ (for appropriate p and q) the results described in Sections 1.1–1.4. We can mimic these results within some Clifford algebras by introducing the concept of spinorial metric.

Observe that since $Spin_+(p, q) \subset \mathbb{R}^+_{p,q}$, it seems interesting to define a scalar product in an ideal $I^+_{p,q} = \mathbb{R}^+_{p,q} e_{pq}$. The reason is that such a scalar product is now *unique*, since if $s \in \mathbb{R}^+_{p,q}$, then $s^* = \tilde{s}$. This unique scalar product will be called in what follows the spinorial metric

$$\beta\colon \quad I^+_{p,q} \times I^+_{p,q} \to F \tag{22}$$

defined by $\beta(\psi, \varphi) = u\tilde{\psi}\varphi$. We see that $G = \{s \in \mathbb{R}^+_{p,q} \mid \tilde{s}s = 1\}$ is the group of automorphisms of the spinorial metric just defined and $G \subset G_1$, $G \subset G_2$.

We now recall the following theorem from Porteous (1981), which says that for $p + q \leq 5$

$$Spin_+(p, q) = \{u \in \mathbb{R}^+_{p,q} \mid \tilde{u}u = u^*u = 1\}$$

With this result we get a new interpretation of the groups $Spin_+(p, q)$ for $p + q \leq 5$, namely, these are the groups that leave the spinorial metric of (22) invariant. But even more important is the fact that now we know how to mimic within appropriate Clifford algebras Sections 1.1–1.4 and thus we can present representations within Clifford algebras of the Pauli c-spinors, undotted and dotted bidimensional c-spinors, and Dirac c-spinors. This is done in Section 4, and in Figueiredo *et al.* (1988).

4. REPRESENTATION OF PAULI c-SPINORS, UNDOTTED AND DOTTED TWO-DIMENSIONAL SPINORS, AND DIRAC c-SPINORS BY APPROPRIATED ALGEBRAIC SPINORS

4.1. Pauli a-Spinors and the Group $SU(2)$

The algebra $\mathbb{R}_{3,0}$ (Pauli algebra) is isomorphic to $\mathbb{C}(2)$ (see Table I). $\mathbb{R}_{3,0}$ is generated by 1 and σ_i, $i=1,2,3$, subject to the condition $\sigma_i\sigma_j+\sigma_j\sigma_i=2\delta_{ij}$, $\delta_{ij}=\text{diag}(+1,+1,+1)$. It is trivial to verify that $e_{30}=1/2(1+\sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$. Now, consider $x\in\mathbb{R}_{3,0}$,

$$x=a_0+a_1\sigma_1+a_2\sigma_2+a_3\sigma_3+a_4\sigma_1\sigma_2+a_5\sigma_1\sigma_3+a_6\sigma_2\sigma_3+a_7\sigma_1\sigma_2\sigma_3 \quad (23)$$
$$a_i\in\mathbb{R}, \qquad i=0,\ldots,7$$

The elements $\varphi\in I_{3,0}\equiv I_p=\mathbb{R}_{3,0}e_{30}=\mathbb{R}^+_{3,0}e_{30}$ (Pauli a-spinors) are of the form

$$\varphi=e_{30}[(a_0+a_3)e_{30}+(a_4+a_7)\sigma_1\sigma_2\sigma_3e_{30}] + \sigma_1e_{30}[(a_1+a_5)e_{30}+(a_2+a_6)\sigma_1\sigma_2\sigma_3e_{30}] \quad (24)$$

It is then immediate that $e_{30}\mathbb{R}_{3,0}e_{30}\simeq\mathbb{C}$ has basis $\{1,\sigma_1\sigma_2\sigma_3\}e_{30}$ and the spinorial basis is $\alpha=\{e_{30},\sigma_1e_{30}\}$. We now show that the elements of I_p are the representatives of Pauli c-spinors (Section 1.1).

Using the isomorphism $\mathbb{R}_{3,0}\overset{f}{\simeq}\mathscr{L}_{\mathbb{C}}(I_p)$ (Section 2.3), $f(x)\psi=x\psi$, $u\in\mathbb{R}_{3,0}$, $\psi\in I_p$, we obtain the representation of $x\in\mathbb{R}_{3,0}$ in the α-basis through the following algorithm.[4] Put

$$e_{30}=|1\rangle, \qquad \sigma_1e_{3,0}=|2\rangle; \qquad \langle 1|=e^*_{30}, \qquad \langle 2|=(\sigma_1e_{30})^*=e_{30}\sigma_1 \quad (25)$$

Then

$$\begin{gathered}1=\sum_i |i\rangle\langle i|, \qquad i=1,2; \qquad \langle i/j\rangle=\delta_{ij}e_{30}\\ x=x\sum|i\rangle\langle i|\Rightarrow x|i\rangle=\sum x_{ji}|j\rangle; \qquad x_{ji}=\langle j|x|i\rangle\\ e_{30}=\begin{bmatrix}1&0\\0&0\end{bmatrix}; \qquad \sigma_1e_{30}=\begin{bmatrix}0&0\\1&0\end{bmatrix};\\ \sigma_1=\begin{bmatrix}0&1\\1&0\end{bmatrix}; \qquad \sigma_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix}; \qquad \sigma_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\end{gathered} \quad (26)$$

We also have the following matrix representations for x, $x^\square$, x^*, and $\tilde{x}\in\mathbb{R}_{3,0}$:

$$\begin{gathered}\mathbb{C}(2)\ni x=\begin{bmatrix}z_1&z_2\\z_3&z_4\end{bmatrix}; \qquad x^\square=\begin{bmatrix}\bar{z}_4&-\bar{z}_3\\-\bar{z}_2&\bar{z}_1\end{bmatrix}\\ x^*=\begin{bmatrix}\bar{z}_1&\bar{z}_3\\\bar{z}_2&\bar{z}_4\end{bmatrix}; \qquad \tilde{x}=\begin{bmatrix}z_4&-z_2\\-z_3&z_1\end{bmatrix}\end{gathered} \quad (27)$$

[4]If $x\in\mathbb{R}_{3,0}$, we use the same letter for $f(x)\in\mathbb{C}(2)\simeq\mathscr{L}_{\mathbb{C}}(I_p)$. This should cause no confusion.

From equations (27) we see that the main antiautomorphism * corresponds in the Pauli algebra to the operation * in matrix algebra.

We now define the spinorial metric

$$\beta:\quad I_p\times I_p\to C;\qquad \beta(\psi,\varphi)=2\langle\psi^*\varphi\rangle_0=\bar{\psi}^1\varphi^1+\bar{\psi}^2\varphi^2 \tag{28}$$

where $\langle\cdot\rangle_0$ means the scalar part of the Pauli number.

The representation of β in the α-basis is then

$$[\beta]_\alpha=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\mathbb{1}_2 \tag{29}$$

Also, $\beta(\psi,\varphi)=\beta(u\psi,u\varphi)\Leftrightarrow u^*u=1\Leftrightarrow u\in U(2)$.

Now, if $x\in\mathbb{R}_{3,0}^+\simeq\mathbb{R}_{0,2}\simeq\mathbb{H}$, we have the following representation for x in the α-basis:

$$x=\begin{bmatrix}z & -\bar{w}\\ w & \bar{z}\end{bmatrix}\qquad\text{and}\qquad \tilde{x}=x^*=\begin{bmatrix}\bar{z} & \bar{w}\\ -w & z\end{bmatrix} \tag{30}$$

Then $N(x)=\tilde{x}x=\det x\cdot\mathbb{1}\Rightarrow N(x)=1\Leftrightarrow\det x=1$. So, the elements $u\in\mathbb{R}_{3,0}^+$ such that $\beta(u\psi,u\varphi)=\beta(\psi,\varphi)$, $\psi,\varphi\in I_p$, satisfy $\tilde{u}u=1$ and $\det u=+1$, which means that $u\in SU(2)\simeq Spin_+(3,0)$. Our statement that Pauli c-spinors are represented by the elements of $I_p=\mathbb{R}_{3,0}^+e_{30}$ (Pauli a-spinors) is then proved.

4.2. Representative of Weyl *c*-Spinors and Dirac *c*-Spinors within $\mathbb{R}_{1,3}$ and $SL(2,\mathbb{C})$

The algebra $\mathbb{R}_{1,3}$ is generated by 1 and the vectors e_μ such that

$$e_\mu e_\nu+e_\nu e_\mu=2\eta_{\mu\nu},\qquad \eta_{\mu\nu}=\mathrm{diag}(+1,-1,-1,-1);\qquad \mu,\nu=0,1,2,3$$

Consider the isomorphism $\mathbb{R}_{3,0}^+\overset{f}{\simeq}\mathbb{R}_{1,3}^+$, where f is the linear extension of $f(\sigma_i)=e_ie_0$ and $\sigma_i\in\mathbb{R}^{3,0}$ as in Section 4.1. Since $e_{30}=1/2(1+\sigma_3)$ is a primitive idempotent of $\mathbb{R}_{3,0}$, $f(e_{30})=1/2(1+e_3e_0)$ is a primitive idempotent of $\mathbb{R}_{1,3}^+$. Also, since $\mathbb{R}_{1,3}\simeq\mathbb{H}(2)$, $e_{13}=f(e_{30})=e$ is also a primitive idempotent of $\mathbb{R}_{1,3}$, since

$$\dim_{\mathbb{R}}\mathbb{R}_{1,3}e=2^4/2\qquad\text{and}\qquad \dim_{\mathbb{H}}\mathbb{R}_{1,3}e=2$$

$I_D=\mathbb{R}_{1,3}e$ is a bidimensional quaternionic space and $\psi_D\in I_D$ is a representation of the Dirac spinors, as we shall prove. Let $a\in\mathbb{R}_{1,3}e$,

$$\begin{aligned}a&=s+(a_0e_0+a_1e_1+a_2e_2+a_3e_3)\\&\quad+(a_{01}e_0e_1+a_{02}e_0e_2+a_{03}e_0e_3+a_{12}e_1e_2+a_{13}e_1e_3+a_{23}e_2e_3)\\&\quad+(a_{012}e_0e_1e_2+a_{013}e_0e_1e_3+a_{023}e_0e_2e_3+a_{123}e_1e_2e_3)\\&\quad+pe_0e_1e_2e_3\end{aligned} \tag{31}$$

Then $\psi_D \in \mathbb{R}_{1,3}e$ is such that

$$\psi_D = e[x_0 e + x_1 e_1 e + x_2 e_2 e + x_3 e_1 e_2 e] + e_1 e_0 e[y_0 e + y_1 e_1 e + y_2 e_2 e + y_3 e_1 e_2 e] \quad (32)$$

where $e\mathbb{R}_{1,3}e \simeq \mathbb{H}$ has basis $\{1, e_1, e_2, e_1e_2\}e$.

4.2.1. Contravariant Undotted a-Spinors

Consider the minimal left ideal $I = \mathbb{R}^+_{1,3}e$. Then $\eta \in I$ can be written as

$$\eta = e(x_0 + x_3\mathbf{i}) + e_0 e_1(y_0 + y_3\mathbf{i}) \quad (33)$$

where $e\mathbb{R}^+_{1,3}e \simeq \mathbb{C}$ has basis $\{1, \mathbf{i}\}e$, where

$$\mathbf{i} \equiv e_5 = e_0e_1e_2e_3 : x_0, x_3, y_0, y_3 \in \mathbb{R}$$

We write

$$\eta = e\eta^1 + e_1 e_0 e\eta^2 = s_1\eta^1 + s_2\eta^2 \simeq \begin{bmatrix} \eta^1 & 0 \\ \eta^2 & 0 \end{bmatrix} \eta^i \in \mathbb{C}e, \; i = 1, 2 \quad (34)$$

4.2.2. Covariant Undotted a-Spinors

Now, the covariant undotted spinors can be identified with the elements of the ideal $\bar{e}\mathbb{R}^+_{1,3} = I_c$, with $\bar{e} = 1 - e$.

Indeed $I_c = (\bar{e}\mathbb{R}^+_{1,3}) = (\mathbb{R}^+_{1,3}e)^{\tilde{}}$ and then if

$$\mathbb{R}^+_{1,3}e \ni \eta = [e(x_0 - x_1\mathbf{i}e) + \sigma_1 e(y_0 e - y_1\mathbf{i}e)] = e\eta^1 + \sigma_1 e\eta^2 \simeq \begin{pmatrix} \eta^1 & 0 \\ \eta^2 & 0 \end{pmatrix}$$

$$\tilde{\eta} = [(x_0\bar{e} - x_1\mathbf{i}\bar{e})\bar{e} - ((y_0\bar{e} - y_1\mathbf{i}\bar{e})\bar{e})\bar{e}\sigma_1] = \eta^1\bar{e} - \eta^2\bar{e}\sigma_1 \simeq \begin{pmatrix} 0 & 0 \\ -\eta_2 & \eta_1 \end{pmatrix}$$

Then

$$\overset{\Delta}{\eta} = \sigma_1\tilde{\eta} = [(x_0 e - x_1\mathbf{i}e)e\sigma_1 - (y_0 e - y_1\mathbf{i}e)e] = (\eta^1 e\sigma_1 - \eta^2 e) \simeq \begin{pmatrix} -\eta^2 & \eta^1 \\ 0 & 0 \end{pmatrix} \quad (35)$$

Note that we can define the spinorial metric is I^c by

$$\beta: \quad I^c \times I^c \to \mathbb{C}; \qquad \beta(\eta, \xi) = 2\langle\overset{\Delta}{\eta}\xi\rangle_0 = \eta^t C\xi \quad (36)$$

4.2.3. Contravariant Dotted Spinors

We identify the contravariant dotted spinors with the elements of the ideal $e\mathbb{R}_{1,3}$. We already know that $e\mathbb{R}_{1,3} \equiv I^{c*}$ and then we identify $\dot{I}^c \equiv I^{c*}$.

Then if $\dot{\eta} \ni \dot{I}^c$ we have

$$\dot{\eta} = [(x_0 e - x_1 \mathbf{i} e)e + (y_0 e - y_1 \mathbf{i} e)e\sigma_1]$$

$$= \dot{\eta}^1 e + \dot{\eta}^2 e\sigma_1 \simeq \begin{pmatrix} \dot{\eta}^1 & \dot{\eta}^2 \\ 0 & 0 \end{pmatrix} \tag{37}$$

4.2.4. Covariant Dotted Spinors

We identify the covariant dotted spinors with the elements of the ideal $\dot{i}_c \equiv \mathbb{R}^+_{1,3}\bar{e} = (e\mathbb{R}_{1,3})^{\tilde{}}$.

Indeed we have that if $\dot{\eta} \in e\mathbb{R}^+_{1,3}$ then

$$\tilde{\dot{\eta}} = [\bar{e}(x_0\bar{e} - x_1\mathbf{i}e) - \sigma_1\bar{e}(y_0 e - y_1 \mathbf{i} e)] \simeq \begin{pmatrix} 0 & -\dot{\eta}^2 \\ 0 & \dot{\eta}^1 \end{pmatrix}$$

Then

$$\tilde{\dot{\eta}}\sigma_1 \equiv [\sigma_1 e(x_0 e - x_1\mathbf{i}e) - e(y_0 e - y_1\mathbf{i}e)] \simeq \begin{pmatrix} -\dot{\eta}^2 & 0 \\ \dot{\eta}^1 & 0 \end{pmatrix} \tag{38}$$

and we put $\overset{\Delta}{\dot{\eta}} = -\tilde{\dot{\eta}}\sigma_1$.

We can define the spinoral metric $\overset{\Delta}{\dot{\beta}}$, by

$$\overset{\Delta}{\dot{\beta}}: \quad \dot{I}^c \times \dot{I}^c \to \mathbb{C}; \qquad \beta(\dot{\eta}, \dot{\xi}) = 2\langle \overset{\Delta}{\dot{\eta}}\dot{\xi}\rangle_0 \tag{39}$$

In the spinorial basis $\alpha = \{e, e_1 e_0 e\}$ of I we have the following representation for $\sigma_i = e_i e_0$, $i = 1, 2, 3$:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{40}$$

Observe now that we can write, from equations (31), (33), and (37), that

$$\psi_D = \varphi + e_0 \overset{\Delta}{\dot{\chi}} \tag{41}$$

Now, let $x \in \mathbb{R}^+_{1,3}$. Then we have that if $x \in \mathbb{C}(2)$ is the representative of $x \in \mathbb{R}^+_{1,3}$, then

$$x = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \Rightarrow e_0 x e_0 = \begin{pmatrix} \bar{z}_4 & -\bar{z}_3 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} = (x^*)^{-1} \tag{42}$$

From equations (41) and (42) we have that, for $u \in Spin_+(1,3)$, $\psi_D \to \psi'_D = u\psi_D$ and

$$u\psi_D = u\varphi + ue_0\overset{\Delta}{\chi} = u\varphi - e_0[(u^*)^{-1}\overset{\Delta}{\chi}] = \varphi' + e_0\overset{\Delta'}{\chi} \tag{43}$$

Equation (41) shows that I_D is the carrier space of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $Spin_+(1,3) \simeq SL(2,\mathbb{C})$ as defined in Section 1.3.

Observe also that from equation (33) we can write

$$\psi_D = e_0 e\psi_1 + e_1 e\psi_2 + e\psi_3 + e_1 e_0 e\psi_4 \tag{44}$$

with $\psi_i \in e\mathbb{R}_{1,3}e \simeq \mathbb{C}$ with basis $\{1, \mathbf{e}\}e_2e_1$.

A complex spinorial basis for I_D is then $\alpha_D = \{e_0e, e_1e, e, e_0e_1e\}$.

Consider now the injection

$$\gamma: \quad \mathbb{R}_{1,3} \to \mathscr{L}_{\mathbb{C}}(I_D)r$$
$$x \mapsto \gamma(x): \quad I_D \to I_D$$
$$\psi_D \mapsto x\psi_D$$

We get the following representation for e_μ, $\mu = 0, 1, 2, 3$, in the α_D-basis:

$$\gamma(e_0) = \gamma_0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \qquad \gamma(e_i) = \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3 \tag{45}$$

We can also mimic the spinorial metric in $\mathbb{C}(4)$ (Section 1.3), defining

$$\beta_D: \quad I_D \times I_D \to \mathbb{C}, \qquad \beta_D(\psi, \varphi) = 2\langle \psi^* e_3 e_1 \varphi \rangle_0 \tag{46}$$

4.3. Representations of Weyl and Dirac *c*-Spinors in $\mathbb{R}_{3,1}$

Now $\mathbb{R}_{3,1} \simeq \mathbb{R}(4)$ (see Table I), the Majorana algebra is generated by 1 and the vectors e_μ such that $\bar{e}_\mu \bar{e}_\nu + \bar{e}_\nu \bar{e}_\mu = -2\eta_{\mu\nu}$; $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, $\mu, \nu = 0, 1, 2, 3$. We can easily verify that $\bar{e} = 1/2(1 + \bar{e}_3\bar{e}_0)$ is a primitive idempotent of $\mathbb{R}^+_{3,1} \simeq \mathbb{R}_{3,0} \simeq \mathbb{R}^+_{1,3}$. Then each $\varphi \in I = \mathbb{R}^+_{3,1}\bar{e}$ can be written as

$$\varphi = \bar{e}\varphi_1 + \bar{e}_1\bar{e}_0\bar{e}\varphi_2 \tag{47}$$

where $\varphi_1, \varphi_2 \in \bar{e}\mathbb{R}^+_{3,1}\bar{e} \simeq \mathbb{C}$ has basis $\{1, \bar{\mathbf{i}}\}\bar{e}$, where $\bar{\mathbf{i}} = -\bar{e}_0\bar{e}_1\bar{e}_2\bar{e}_3 = -\sigma_1\sigma_2\sigma_3$, $\sigma_i = \bar{e}_i\bar{e}_0$, $i = 1, 2, 3$.

It follows that the structure of the Weyl spinors is equal in the $\mathbb{R}_{1,3}$ algebra. What we want now is to represent the Dirac spinors inside $\mathbb{R}_{3,1}$. Observe that unlike the case of $\mathbb{R}_{1,3}$, $\bar{e} = 1/2(1 + \bar{e}_3\bar{e}_0)$ is not a primitive idempotent of $\mathbb{R}_{3,1}$.

However, for each $x \in \mathbb{R}_{3,1}$ we can write

$$\mathbb{R}_{1,3} \ni x = x^+ + \bar{e}_0 y^+; \qquad x^+, y^+ \in \mathbb{R}^+_{3,1} \tag{48}$$

Also, if $u \in \mathbb{R}^+_{3,1}$ and if $u \in \mathbb{C}(2)$ is the representative of u in the canonical spinorial basis, then

$$u = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \Rightarrow \bar{e}_0 u \bar{e}_0 = \begin{pmatrix} -\bar{z}_4 & \bar{z}_3 \\ \bar{z}_2 & -\bar{z}_1 \end{pmatrix} = -(u^*)^{-1} \tag{49}$$

It follows that the objects of the nonminimal ideal

$$\bar{I}_D = \mathbb{R}_{1,3}\bar{e}, \qquad \psi_D = \varphi + \bar{e}_0 \overset{\Delta}{\dot{\chi}}$$

transforms under the action of $u \in Spin_+(3,1) \simeq Spin_+(1,3) \simeq SL(2, \mathbb{C})$ as

$$\bar{\psi}_D \to u\bar{\psi}_D = u\varphi + u\bar{e}_0 \overset{\Delta}{\dot{\chi}} = u\varphi + \bar{e}_0[(u^*)^{-1} \overset{\Delta}{\dot{\chi}}] \tag{50}$$

From equation (50) it follows that the nonminimal ideal $\bar{I}_D$ is the carrier space of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of the group $SL(2, \mathbb{C})$. The $\bar{\psi}_D$ is an e-spinor according to the definition given in Section 3.2.

This example shows that when working with Clifford algebras we cannot restrict the representation of the c-spinors used by physicists *only* to elements of minimal lateral ideals.

4.3.1. *Majorana Spinors*

The elements of the minimal left ideals of $\mathbb{R}_{3,1}$ are the Majorana spinors. They can be constructed by the standard procedure used above.

Since $\hat{e} = 1/4(1 + \bar{e}_3\bar{e}_0)(1 + \bar{e}_2)$ is a primitive idempotent of $\mathbb{R}_{3,1}$, we have

$$I_M = \mathbb{R}_{3,1}\hat{e} \ni \psi_M = \bar{e}_1\hat{e}\psi_1 + \hat{e}\psi_2 + \bar{e}_0\bar{e}_1\hat{e}\psi_3 + \bar{e}_0\hat{e}\psi_4 \tag{51}$$

where $\psi_i \in \hat{e}\mathbb{R}_{3,1}\hat{e} \simeq \mathbb{R}$, $i = 1, 2, 3, 4$.

It is interesting to compare (51) with (44), which express ψ_M in I_M and ψ_D in I_D.

Consider now the isomorphism

$$\gamma\colon\quad \mathbb{R}_{3,1} \to \mathscr{L}_{\mathbb{R}}(I_M)$$

$$x \mapsto \gamma(x)\colon\quad I_M \to I_M$$

$$\psi_M \mapsto x\psi_M$$

We get the following representation for $\bar{e}_\mu$, $\mu = 0, 1, 2, 3$, in the $\alpha_M = \{\bar{e}_1, \hat{e}, \bar{e}_1, \bar{e}_0\bar{e}_1\hat{e}, \bar{e}_0\hat{e}\}$ basis $[\gamma(e_\mu) \equiv \gamma_\mu]$,

$$\gamma_0 = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}; \quad \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}; \quad \gamma_2 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}; \quad \gamma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \tag{52}$$

Equations (52) show a set of what is usually called in the literature Majorana matrices and our presentation shows how easy it is to find a set of Majorana matrices with the techniques of this paper.

4.4. Representations of Dirac *c*-Spinors within the $\mathbb{R}_{4,1}$ Algebra

From Table I we see that $\mathbb{R}_{4,1}$, $\mathbb{R}_{2,3}$, and $\mathbb{R}_{0,5}$ are isomorphic to the algebra $\mathbb{C}(4)$, which is the usual Dirac algebra of physicists. In order to identify the algebra that carries the physical interpretation associated with space-time ($\mathbb{R}^{1,3}$), we proceed as follows. Let E_A, $A = 0, 1, 2, 3, 4$, be an orthonormal basis for $\mathbb{R}^{p,q}$ with $p + q = 5$. The volume element is $E_J = E_0E_1E_2E_3E_4$ and we get $E_J^2 = -1$ for $q = 1, 3, 5$. Now define

$$e_\mu = E_\mu E_4 \tag{53}$$

and impose that e_μ is an orthonormal basis for $\mathbb{R}^{1,3}$, i.e.,

$$e_0^2 = -E_0^2E_4^2 = +1, \qquad e_k^2 = -E_k^2E_4^2 = -1, \qquad k = 1, 2, 3 \tag{54}$$

Equations (54) are satisfied when $p = 4$, $q = 1$, i.e., $E_4^2 = E_k^2 = -E_0^2 = 1$, and we conclude that the real Clifford algebra associated with space-time ($\mathbb{R}^{1,3}$) and isomorphic to $\mathbb{C}(4)$ is $\mathbb{R}_{4,1}$.

Equation (54) shows that $\mathbb{R}_{1,3} \stackrel{g}{\simeq} \mathbb{R}_{4,1}^+$ where g is the linear extension of $g(e_\mu) = E_\mu E_4$, $\mu = 0, 1, 2, 3$. We already saw in Section 4.2 that $f(e_{30})$ is a primitive idempotent of $\mathbb{R}_{1,3}$ and we have that $g(f(e_{30}))$ is a primitive idempotent of $\mathbb{R}_{4,1}^+$. Then $\bar{I}_D = \mathbb{R}_{4,1}^+ g(f(e_{30}))$ is a minimal ideal $\mathbb{R}_{4,1}^+$ which is a 4-dimensional vector space over the complex field and its elements, the Dirac *a*-spinors, are representations in $\mathbb{R}_{4,1}$ of Dirac *c*-spinors.

4.5. Representation of the Standard Dirac *c*-Spinors within the $\mathbb{R}_{1,3}$ Algebra

It is obvious that $\check{e} = 1/2(1 + e_0)$ is a primitive idempotent of $\mathbb{R}_{1,3}$. Also we can easily verify that for any $x \in \mathbb{R}_{1,3}$ there exists $y \in \mathbb{R}_{1,3}^+$ such that $x\check{e} = y\check{e}$. It follows that $\check{I}_D = \mathbb{R}_{1,3}^+\check{e}$ is a minimal left ideal of $\mathbb{R}_{1,3}$. The elements $\check{\psi}_D \in \check{I}_D$

can be written in the form

$$\check{\psi} = \check{e}\psi_1 + e_3 e_1 \check{e}\psi_2 + e_3 \check{e}\psi_3 + e_1 \check{e}\psi_4 \tag{55}$$

$\psi_i \in \check{e}\mathbb{R}_{1,3}\check{e} \simeq \mathbb{C}$ with basis $\{1, e_2 e_1\}\check{e}$.

The $\check{\psi}$ are the representatives of standard Dirac c-spinors, which are the kind of c-spinors that appear in the usual form of the Dirac equation (Landau and Lifschitz, 1971).

The isomorphism

$$\begin{aligned} \gamma:\quad & \mathbb{R}_{1,3} \to \mathscr{L}_{\mathbb{C}}(\check{I}_D) \\ & x \mapsto \gamma(x): \quad \check{I}_D \to \check{I}_D \\ & \qquad\qquad\qquad \check{\psi} \mapsto x\check{\psi} \end{aligned}$$

gives, through the technique introduced in Section 4.1, the following representation for e_μ, $\mu = 0, 1, 2, 3$, and $e_5 = e_0 e_1 e_2 e_3$ in $\check{\alpha}_D = \{\check{e}, e_3 e_1 \check{e}, e_3 \check{e}, e_1 \check{e}\}$, a complex spinorial basis for $\check{I}_D$.

Putting $\gamma(\check{\alpha}_D) = \{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$ and $\gamma(e_\mu) = \gamma_\mu$, $\gamma(e_5) = \gamma_5$, we have

$$i = [\delta_{i1}]; \qquad \gamma_0 = \begin{bmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{bmatrix}; \qquad \gamma_k = \begin{bmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{bmatrix}; \qquad \gamma_5 = \begin{bmatrix} 0 & i\mathbb{1}_2 \\ i\mathbb{1}_2 & 0 \end{bmatrix} \tag{56}$$

where $[\delta_{i1}]$ is the 4×4 matrix with one in the i-line of the first column, all other elements being zero. Also, $i = \sqrt{-1}$. The set of γ matrices in (56) is usually known as the standard representation of Dirac matrices (Landau and Lifschitz, 1971).

Observe now that the idempotents $\check{e} = 1/2(1 + e_0)$ and $e = 1/2(1 + e_3 e_0)$ are related by

$$e = u\check{e}u^{-1}; \qquad u = (1 + e_3) \tag{57}$$

Since $u = (1 + e_3) \notin \Gamma(1, 3)$, the ideals I_D and $\check{I}_D$ are not equivalent (module) representations of $\mathbb{R}_{1,3}$ (Figueiredo *et al.*, 1988), although from the point of view of group theory both ideals are carrier spaces of the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2, \mathbb{C})$. The point is important for papers II and III of this series. There we will need also the following results, which are trivially established:

$$\begin{gathered} |1\rangle = \gamma_0|1\rangle; \qquad i|1\rangle = \gamma_2\gamma_1|1\rangle; \qquad |2\rangle = -\gamma_5\gamma_2|1\rangle; \\ |3\rangle = \gamma_3|1\rangle; \qquad |4\rangle = \gamma_1|1\rangle \end{gathered} \tag{58}$$

4.6. Operator Spinors

Hestenes (1967, 1971*a*,*b*, 1975; Hestenes and Sobezyk, 1984) defines an operator (*o*-spinor) in $\mathbb{R}_{p,q}$ as follows:

An *o*-spinor in $\mathbb{R}_{p,q}$ is a bilinear map $\mathbb{R}^+_{p,q} \ni \psi: \mathbb{R}^{p,q}$ given by $\psi x \psi^{-1} = y$ for $x, y \in \mathbb{R}^{p,q}$.

Now, the invertible elements of $\mathbb{R}_{p,q}$ such that $u^{\square} x u^{-1} = y$, for $x, y \in \mathbb{R}^{p,q}$, $u \in \mathbb{R}_{p,q}$, form the so-called Clifford group $\Gamma(p, q)$ of $\mathbb{R}_{p,q}$ (Section 3.1). The special Clifford group $\Gamma^+(p, q)$ is defined by $\Gamma^+(p, q) = \Gamma(p, q) \cap \mathbb{R}^+_{p,q}$. [For more details see Figueiredo *et al.* (1988).]

We then immediately recognize *o*-spinors as elements of $\Gamma^+(p, q)$. If we remember now that $Spin_+(p, q) = \{u \in \Gamma_+(p, q) | N(u) = +1\}$, we see that we can write $U^{\square} x U^{-1} = U x U^{-1}$, $\forall x \in \mathbb{R}^{p,q}$, $\forall U \in Spin_+(p, q)$. Then for each $\psi \in \Gamma^+(p, q)$ we can write

$$\psi x \psi^{-1} = y = \rho z; \qquad x, y, z \in \mathbb{R}^{p,q}; \qquad \rho \in \mathbb{R}; \qquad x^2 = z^2 \tag{59}$$

Then,

$$\psi x \psi^{-1} = \rho u x u^{-1} \tag{60}$$

Solving for ρx, we get $\lambda x \lambda^{-1} = \rho x$ with $\lambda = u^{-1}\psi$. This last equation has a solution $\forall x \in \mathbb{R}^{p,q}$ only if λ has a scalar and possibly a pseudoscalar part. We can show that the pseudoscalar part vanishes unless $p + q = 4m$, where m is an integer.

For $\mathbb{R}_{1,3}$ we see that the *o*-spinor can be written as a sum of two even multivectors and has eight degrees of freedom, since $\psi = u^{-1}\lambda = u^{-1}(a + e_5 b)$, where $a, b \in \mathbb{R}$ and e_5 is the unit pseudoscalar of $\mathbb{R}_{1,3}$. Thus, an *o*-spinor of $\mathbb{R}_{1,3}$ has the same degrees of freedom as a covariant (or algebraic) Dirac spinor.

The correspondence between a Dirac *o*-spinor and a Dirac *a*-spinor is evident from Section 4.5, since a Dirac *a*-spinor is the product of an even multivector by the particular idempotent $e = \frac{1}{2}(1 + e_3 e_0)$.

To end this section we remark that the above correspondence, *a*-spinors $\leftrightarrow$ *o*-spinors, is simply a generalization of the well-known result that we can associate vectors in $\mathbb{R}^{p,q}$ with elements $\sigma \in SO_+(p, q)$ and covariant spinors associated with $\mathbb{R}^{p,q}$ with the elements of the $Spin_+(p, q)$ group (Bleecker, 1981). The result for $p + q + \leq 5$ is even more justified since in this case $Spin_+(p, q) = \{u \in \mathbb{R}^+_{p,q} \,|\, \tilde{u}u = 1\}$.

5. CONCLUSIONS

Hestenes (1986) said about the theory of spinors: "I have not met anyone who was not dissatisfied with his first reading on the subject."

Well, the reasons for such statement are in our view due to three main facts:

(A) The usual representation of *c*-spinors such as introduced in Sections 1.1–1.4 does not emphasize the geometrical meaning of these objects.
(B) There are no clear connections between the abstract concepts of *c*-spinors and the more abstract concepts of *a*-spinors, *e*-spinors, and *o*-spinors as elements of particular Clifford algebras.
(C) The representation of *a*-spinor (or *e*-spinor) fields as sections of some Clifford bundles (over space-time) and the problem of the "transformation law" of spinors.

As to (A), we think that the situation has been partially clarified with the presentation by Hestenes (1967, 1971*a*,*b*, 1975; also see Lounesto, 1986) of the geometrical meaning of Pauli spinors and of Dirac spinors and also by Penrose and Rindler (1984) of the quasigeometrical representation of the undotted and dotted two-component spinors.

As to (B), we think that the present paper shows in a clear way how to obtain relations between all *c*-spinors used by physicists and *a*-spinors and *e*-spinors. The relation of *o*-spinors is also clarified in Section 4.

From our approach, in Sections 3.5 and 4 it is clear that *a*-spinors and *e*-spinors can be thought of as elements of the exterior algebra of the vector space $V=\mathbb{R}^{p,q}$. It follows that the usual claim that spinors are more fundamental than tensors is a *non sequitur.*

It is very important to emphasize that all our *a*-spinors or (*e*-spinors) are elements of *real Clifford algebras.* Other approaches to the subject of algebraic spinors (e.g., Crumeyrolle, 1969, 1971; Bugajska, 1979; Salingrados and Wene, 1985) complexify $\mathbb{R}_{1,3}$ or $\mathbb{R}_{3,1}$ [the complexification being isomorphic to $\mathbb{R}_{4,1}\simeq\mathbb{C}(4)$], introducing unnecessary complications. The reason for such a complexification is the need to use a de Witt (Crumeyrolle, 1969, 1971; Bugajska, 1979) basis for the spinor space, since those authors seemed unaware of the idempotent method used in this paper.

Another "need" for complexification comes, according to the view of Salingrados and Wene (1985) from the fact that $\mathbb{R}_{1,3}$ has only two idempotents and the formulation of quantum electrodynamics, as is well known, needs four idempotents (there called projection operators). This "difficulty" can be easily solved following Hestenes (1986) simply by introducing a single operator that belongs to the dual space $\mathbb{R}_{1,3}$ (here considered as a vector space over $\mathbb{R}$).

We cannot properly discuss here some distinctive features of the different representations of Dirac *a*-spinor (or *e*-spinor) fields and related

Dirac-like equations over Lorentzian manifolds. The interested reader should see papers II and III and also Figueiredo *et al.* (1988).

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RERERENCES

Atiyah, M. F., Bott, R., and Shapiro, A. (1984). Clifford modules, *Topology* **3**(Supp. 1), 3.

Benn, I. M., and Tucker, R. W. (1983*a*). *Physics Letters*, **130B**, 177.

Benn, I. M., and Tucker, R. W. (1983*b*). Fermions without spinors, *Communications in Mathematical Physics*, **89**, 34.

Benn, I. M., and Tucker, R. W. (1983*b*). A local right-spin covariant Kähler equation, *Physics Letters*, **130B**, 177.

Benn, I. M., and Tucker, R. W. (1985*a*). The differential approach to spinor and their symmetries, *Nuovo Cimento*, **88A**, 273.

Benn, I. M., and Tucker, R. W. (1985*b*). The Dirac equation in exterior form, *Communications in Mathematical Physics*, **98**, 53.

Bichteler, K. (1963). Global existence of spin structures for gravitational fields, *Journal of Mathematical Physics*, **9**, 198.

Blaine Lawson, Jr., H., and Michelsohn, M. L. (1983). *Spin Geometry*, Universidad Federal do Ceará, Brazil.

Blau, M. (1987). Connections on Clifford bundles and the Dirac operator, *Letters in Mathematical Physics*, **13**, 83.

Bleecker, D. (1981). *Gauge Theory and Variational Principles*, Addison-Wesley, Reading, Massachusetts.

Brauer, R., and Weyl, H. (1935). Spinors in *n* dimensions, *American Journal of Mathematics*, **57**, 425.

Budinich, P., and Trautman, A. (1986). Remarks on pure spinors, *Letters in Mathematical Physics*, **11**, 315.

Bugajska, B. (1979). Spinor structure of space-time, *International Journal of Theoretical Physics*, **18**, 77.

Caianello, E. R. (1988). "Spineurs Simples", "Urfelder" and factorizations of Dirac equations and spinors, *Physica Scripta*, **37**, 197.

Cartan, E. (1966). *Theory of Spinors*, Dover, New York.

Chevalley, C. (1954). *The Algebraic Theory of Spinors*, Columbia University Press, New York.

Coquereaux, R. (1982). Modulo 8 periodicity of real Clifford algebras and particle physics, *Physics Letters B*, **115**, 189.

Crumeyrolle, A. (1969). Structures spinorielles, *Annales de l'Institut Henri Poincarè, A*, **XI**(1), 19.

Crumeyrolle, A. (1971). *Annales de l'Institut Henri Poincarè*, **14**, 309.

Dimakis, A. (1986). In *Clifford Algebras and their Applications to Mathematical Physics* J. S. R. Chisholm and A. K. Common, eds.), D. Reidel, Dordrecht.

Felzenswalb, B. (1979). *Algebras de Dimensão Finita*, Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brazil.

Figueiredo, V. L., de Oliveira, E. C., Rodrigues, Jr., W. A. (1990). Clifford Algebras and the hidden geometrical nature of spinors, *Hadronic Journal*, to appear.

Geroch, R. (1968). Spinor structure of space-times in general relativity I, *Journal of Mathematical Physics*, **9**, 813.

Graf, W. (1978). Differential forms as spinors, *Annales de l'Institut Henri Poincarè*, **XXIX**, 85.

Hestenes, D. (1967). Real spinor fields, *Journal of Mathematical Physics*, **8**, 798.

Hestenes, D. (1971*a*). Vectors, spinors, and complex numbers in classical and quantum physics, *American Journal of Physics*, **59**, 1013.

Hestenes, D. (1971*b*). Local observables in quantum theory, *American Journal of Physics*, **39**, 1028.

Hestenes, D. (1975). Observables, operators, and complex numbers in the Dirac theory, *Journal of Mathematical Physics*, **16**, 556.

Hestenes, D. (1986). In *Clifford Algebras and Their Applications to Mathematical Physics*, J. S. R. Chisholm and A. K. Common, eds., D. Reidel, Dordrecht.

Hestenes, D., and Sobezyk, G. (1984). *Clifford Algebra to Geometrical Calculus*, D. Reidel, Dordrecht.

Landau, L. D., and Lifschitz, E. M. (1971). *Relativistic Quantum Theory*, Addison-Wesley, Reading, Massachusetts.

Lichnerowicz, A. (1964). Champs Spinoriels et Propagateurs en Relativité Générale, *Bulletin Société Mathématique France* **92**, 11.

Lounesto, P. (1981). Scalar products of spinors and an extension of Brauer–Wall groups, *Foundations of Physics*, **11**, 721.

Lounesto, P. (1986). In *Clifford Algebras and Their Applications in Mathematical Physics*, J. S. R. Chisholm and A. K. Common, eds., D. Reidel, Dordrecht.

Micali, A. (1986). In *Clifford Algebras and Their Applications in Mathematical Physics*, J. S. R. Chisholm and A. K. Common, eds., D. Reidel, Dordrecht.

Miller, W., Jr. (1972). *Symmetry Groups and Their Applications*, Academic Press, New York.

Milnor, J. W. (1963). Spin structures on manifolds, *L'Enseignement Mathematique*, **9**, 198.

Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, Vols. I and II, Cambridge University Press, Cambridge.

Porteous, I. R. (1981). *Topological Geometry*, 2nd ed., Cambridge University Press, Cambridge.

Riesz, M. (1958). *Clifford Numbers and Spinors*, Lecture Notes No. 38, Institute for Fluid Mechanics and Applied Mathematics, University of Maryland.

Rodrigues, Jr., W. A., and de Oliveira, E. C. (1990). Dirac and Maxwell equations in the Clifford and spin Clifford bundles, *International Journal of Theoretical Physics*, this issue.

Rodrigues, Jr., W. A., and Figueiredo, V. L. (1989). In *Proceedings VIII Convegno Nazionale di Relatività Generale e Fisica della Gravitazione*, M. Toller, M. Cerdonio, M. Francaviglia, and R. Cianci, eds., World Scientific, Singapore.

Rodrigues, Jr., W. A., and Figueiredo, V. L. (1990). Real spin-Clifford bundle and the spinor structure of space-time, *International Journal of Theoretical Physics*, this issue.

Salingaros, N. A., and Wene, G. P. (1985). The Clifford algebra of differential forms, *Acta Applicadae Mathematica*, **4**, 271.

Santaló, L. A. (1976). *Geometria Espinorial*, Consejo Nacional de Inv. Cient. y Tecnica, Int. Argentino de Matematica, Buenos Aires, Argentina.

Srivastrava, P. P. (1974). On spinor representation of the Lorentz group, *Revista Brasileira de Fisica*, **4**, 507.

Van der Waerden, B. L. (1932). *Group Theory and Quantum Mechanics*, Springer, Berlin.

Weyl, H. (1929). Elektron and Gravitation 1.2, *Physik*, **56**, 330.